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# MODIFIED TRANSFER EQUATION APPROACH AND ITS APPLICATION TO LINEARIZED PLANE COUETTE FLOW

by

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## *1. Introduction:*

Many practical problems in the theory of rarefied gases become susceptible to mathematical treatment through the transfer equation approach. In this approach an approximate form of the velocity distribution function containing a number of coefficients with spatial and temporal variations is assumed whereas the coefficients are determined by satisfying judiciously selected transfer equations. These equations are formed by multiplying the collision equations by certain factors, e.g. the mass of a molecule, the momentum of a molecule, etc. and then to integrate the equation over all values of the velocity components of the molecule. Major contributions for the advancement of the transfer equation approach have been made by Mott-Smith (1951)[1], Lees (1959)[3], Takao (1961)[5], Shen (1963)[6] and Beck (1965)[7]. For completeness and convenience we discuss here only the main points of each of the methods proposed by these authors before dealing with the present method, viz. "Modified Transfer Equation Approach".

Mott-Smith (1951)[1] approximates the distribution function by a sum of two Maxwellian distribution functions corresponding to the conditions upstream and downstream of the shock. The velocity and temperature functions in these Maxwellians are taken as constants and are therefore related by the usual Rankine-Hugoniot conditions, but the number densities of these Maxwellians are initially undetermined. The number density distribution through the shock region is found by solving the pertinent transport equations. This method is quite suitable for a rough description of the structure of a strong, steady normal shock wave. In this connection we also mention a slightly different approach initiated by Glansdorff (1961)[2] which consists in considering the gas a mixture of an upstream and a downstream gas both in self-equilibrium, but experiencing inelastic collision from each other. It is believed to retain the virtues of Mott-Smith approach in case of a strong shock and it improves it in case of weak shock if the Navier-Stokes structure is taken as reference. It is in fact an extension to the gas dynamics of the two fluid model initially introduced in the superfluidity theory. In a two fluid model each component keeps its own temperature and its own mean velocity, contrary to the usual diffusion theory. Lees (1959)[3] introduces the two stream Maxwellian which is a natural extension and generalization of Mott-Smith's function (1951)[1] for a normal shock, but differs from it in certain essential respects. One important difference between Lees' formulation and Mott-Smith's is that the distribution functions employed by Lees are discontinuous in particle velocity, whereas Mott-Smith's function is continuous. It should be remarked, however, that Lees' approach seems to have been inspired by Mott-Smith's treatment of the normal shock wave. Lees applies his technique to linearized plane couette flow and Rayleigh's problem and to that end he utilizes two stream Max-

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wellian in equations of transfer. Further, Liu and Lees (1961)[4] extend the technique of Lees (1959)[3] to cover the problem of plane compressible couette flow.

By using the line of sight principle postulated by Lees (1959)[3], which in fact divides the molecules into groups as if they were in free-molecule flow, a more general formulation for the velocity distribution functions (viz. the bimodal two stream Maxwellian distribution) is proposed by Beck (1965)[7]. The assumption of Lees for the velocity distribution function is contained as a special case in the formulation of Beck.

Shen (1963)[6] proposes a relaxation type procedure for the transition regime of Rarefied-gas Flows and applies it to the problem of couette flow. He claims "though more cumbersome than earlier, simpler approaches, the present theory is fundamentally more defensible and seems to yield better details in Kundsen layer". He makes use of Lees' two stream Maxwellian in his assumed form of distribution function. This point is dealt with in the section 2. The present author believes that instead of using Lees' two-stream Maxwellian distribution it is more logical to use polynomials in the velocity space. In Shen's approach one can only choose "appropriate" transfer equations so as to calculate the unknowns introduced in the distribution function, but in the present analysis one can choose as many transfer equations as one requires. Moreover, the series expansion of distribution function has more variables which are functions of position and can be evaluated by the appropriate (unrestricted) number of transfer equations whereas in Shen's procedure there are only two unknowns. Therefore the present method is more general than that of Shen's and can take full advantage of the additional transfer equations.

In short it is the purpose of the present paper to propose a more general expression for the distribution function than that of Shen's and the application of which is made to the problem of plane linearized couette flow. The unknowns introduced in the distribution function are determined by means of transfer equations. We will designate the present approach as "Modified Transfer Equation Approach".

In the following section the main features of the present method are exposed.

## 2. *Basic Theory of the Present Method.*

In studying the problems of rarefied gas dynamics, one must generally have recourse to the Boltzmann equation. For the general case one has a non-linear, integro-differential equation, the integral involving a fivefold integration. In the linear problem one can simplify this integral by linearization and reduce the order of integration from five to three; however, even for this simplified case, the integro-differential equation presents a formidable mathematical difficulty. To treat this problem various investigators Bhatnagar et al (1954)[8] and Welander (1954)[9], have postulated models which replace the complicated collision integrals and which is

$$\left(\frac{\partial}{\partial t} + \underline{v}.\nabla\right) nf = -\nu(nf - n^{(o)} f^{(o)}) \tag{1}$$

where

$f = f(t, \underline{x}, \underline{v})$, velocity distribution function,

$\underline{v} = (v_x, v_y, v_z)$, molecular velocity,

$$n^{(o)} f^{(o)} = n(\underline{x}, t) \left[\frac{m}{2\pi k T(\underline{x}, t)}\right]^{3/2} \exp\left[-\frac{m}{2kT(\underline{x}, t)}\left\{\underline{v} - \underline{q}(\underline{x}, t)\right\}^2\right],$$

local Maxwellian

$$n(\underline{x}, t) = \int f d\underline{v}, \quad \text{number density}$$

$$\underline{q}(\underline{x}, t) = n^{-1} \int \underline{v} f d\underline{v}, \quad \text{macroscopic velocity}$$

$$T(\underline{x}, t) = \frac{m}{3n(\underline{x}, t)k} \int \left\{ \underline{v} - \underline{q}(\underline{x}, t) \right\}^2 d\underline{v}, \quad \text{temperature}$$

and $\nu$ is the collision frequency.

The BGK model is based on the assumption that the molecules scattered into the beam of molecules come from collision between molecules into Maxwellian motion. This model equation exhibits all the elementary qualitative features of the Boltzmann Equation.
Under steady state conditions

$$\frac{\partial f}{\partial t} = 0$$

Thus after taking cognizance of the fact that $n^{(o)} f^{(o)}$ must be constant in this case and ergo upon integration, eq.(1) results as:

$$nf = n'f' \exp\left[ - \nu(s-s')/v \right] + n^{(o)} f^{(o)} \left[ 1 - \exp\left\{ -\nu(s-s')/v \right\} \right] \qquad (2)$$

where s, v, n'f' represent the distance measured along the direction of $\underline{v}$, the magnitude of the velocity $\underline{v}$ and the distribution prescribed at the place of origin (s=s') of the molecule having velocity $\underline{v}$ respectively. It should be remarked that the assumed distribution function forms the starting point of the transfer equation approach. Among the investigators employing the transfer equation approach, only Shen indicates how an approximate distribution function may be constructed taking into account the proper collision effects. For this, Shen examines the Krook equation (1) for simplicity and proposes the approximate distribution function (3) on the basis of equation (2), viz.

$$nf = n'f' \exp\left[ - k(s-s')/v \right] + n^*f^* \left[ 1 - \exp\left\{ - k(s-s')/v \right\} \right], \qquad (3)$$

where the distribution function $n^*f^*$ is not the same as the local Maxwellian $n^{(o)} f^{(o)}$. Shen takes $n^*f^*$ as Lees type of distribution function with limited number of unknowns and his approach does not exploit the full use of transfer equations. Therefore it becomes expedient to introduce the series type expansion for $n^*f^*$ with as many unknowns as possible which can be evaluated with the aid of transfer equations. In a sense the present method, inspite of the same amount of work involved, is more general than that of Shen. The above procedure is applied to linearized couette flow and the analysis is outlined in the following section.

## *3. Modified Transfer Equation Method for Linearized Plane Couette Flow*

The present approach is applied to the problem of steady flow of gas between two parallel plates. The upper plate moves with velocity + u/2 in its own plane at y = d/2, while the lower plate at y = - d/2 moves parallel to the upper plate with velocity - u/2. Both the temperature and the density $\rho$(hence the number density n) are constants for the linearized case. Further it is necessary to select a model for the interaction of the molecules with these plates, and for simplicity it is assumed that the molecules reflect with Maxwell equilibrium distribution specified by the plate temperature, i.e. diffuse reflection with complete thermal accommodation. Hence on leaving the plates y = ± d/2, the molecules have the Maxwellian

distribution:

$$nf = nA \exp\left\{- h\left[\left(\xi_x \mp \frac{u}{2}\right)^2 + \xi_y^2 + \xi_z^2\right]\right\} \tag{4}$$

with

$$A = \left(\frac{\pi}{h}\right)^{-3/2} \tag{5}$$

and $h = \frac{1}{2RT}$

The upper sign in (4) holds for plate "1" while the lower sign for plate "2".

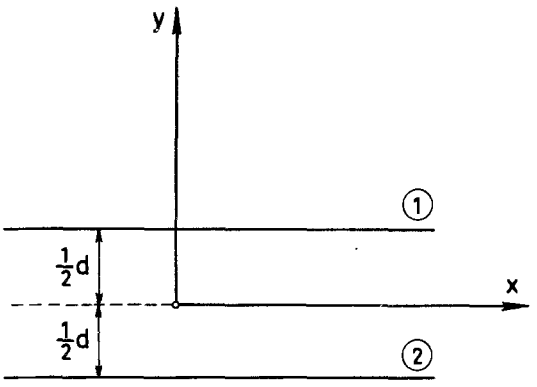


Fig. 1. The Linearized Couette Flow.

The only independent variable in this problem is the coordinate normal to the plate; thus (3) transforms to:

$$nf = n'f' \exp\left[\frac{-k(y-y')}{\xi_y}\right] + n^*f^* \left\{1 - \exp\left[\frac{-k(y-y')}{\xi_y}\right]\right\}, \tag{6}$$

where $\xi_y$ is the component of molecular velocity in the y-direction and y' indicates the place of the origin of the molecules. As the choice of n*f* is arbitrary and further if the distribution function is chosen so as to satisfy number of transport equations, one can reach at reasonably accurate approximation of the distribution function rather than having very limited number of transfer equations - a result of assumed form of distribution function having limited number of unknowns. In the present method we assume the distribution function for n*f* a series expansion in velocity space as follows:

$$\xi_y > 0,$$

$$nf = nA\exp\left\{- h\left[\left(\xi_x + \frac{u}{2}\right)^2 + \xi_y^2 + \xi_z^2\right] - \frac{k(y+1)}{\xi_y}\right\} +$$

$$+ nA\left[1 - \exp\left(\frac{-k(y+1)}{\xi_y}\right)\right] \exp\left\{- h\xi^2\right\}\left[\alpha_1^+(y)\xi_x + \alpha_2^+(y)\xi_y + \alpha_3^+(y)\xi_x\xi_z + \ldots\right] \tag{7}$$

$$\xi_y < 0$$

$$nf = nA\exp\left\{ - \left[ \left(\xi_x - \frac{u}{2}\right)^2 + \xi_y^2 + \xi_z^2 \left(\frac{-k(y-1)}{\xi_y}\right) \right] \right\} +$$

$$+nA\left[1 - \exp\left(\frac{-k(y-1)}{\xi_y}\right)\right] \exp\left\{-h\xi^2\right\} \left[\alpha_1^-(y)\xi_x + \alpha_2^-(y)\xi_y + \ldots\ldots\right] \qquad (7)$$

where $\alpha_i^{\mp}(y)$ are functions to be determined by the appropriate transfer equations. The boundary conditions are automatically satisfied by (7). Moreover, in the preceding analysis we have assumed d=2 for the sake of simplicity. In order to calculate $\alpha_i^{\mp}(y)$ we make use of the transfer equations and first consider the equations for $m\xi_x$ and $m\xi_x\xi_y$. It is worth remarking that while considering the transfer equations for the above said variables all the terms except with coefficients $\alpha_1^{\mp}(y)$ vanish as a result of the form of the distribution functions (7). Moreover, if we choose other transfer equations e.g. for $m\xi_y$ and $m\xi^2{}_y$, same sort of simplification is bound to result. It is evident that on account of the broad choice of transfer equations, one may be able to span the wider spectrum of non-linear problems. Without any loss of generality, we will calculate only four of the variables introduced in the distribution functions and make use of four moment equations in order to calculate them.

*a. Calculation of* $\alpha_1^{\mp}(y)$. The transfer equations for $m\xi_x$ and $m\xi_x\xi_y$ yield the following two equations for $\alpha_1^{\mp}(y)$, viz.:

$$\frac{\alpha_1^+}{4\sqrt{\pi}h^{3/2}}(1-G_{1,+}) - \frac{\alpha_1^-}{4\sqrt{\pi}h^{3/2}}(1-G_{1,-}) = \frac{1}{\rho}p_{xy} + \frac{u}{4\sqrt{\pi}h}(G_{1,+}+G_{1,-}), \qquad (8)$$

$$\frac{\alpha_1^+}{8h^2}(1-G_{2,+}) + \frac{\alpha_1^-}{8h^2}(1-G_{2,-}) = \frac{RT}{\mu}p_{xy}\,(y+C_1) + \frac{u}{8h}(G_{2,+} - G_{2,-}), \qquad (9)$$

where $G_{n,\pm} = G_n\left[k\sqrt{h}\,(1\pm y)\right]$,

$$\overline{G_n(\alpha)} = \frac{\int_0^\infty \exp(-x^2-\alpha/x)x^n dx}{\int_0^\infty \exp(-x^2)x^n dx}, \quad n = 0, 1, 2, \ldots,$$

and the calculation of $G_n(\alpha)$ is discussed by Abramovitz (1953)[10]. It is interesting to note, however, that the equations for $\alpha_1^{\mp}(y)$ turn out to be of the similar nature as that for U in case of Shen. Before determining $\alpha_1^{\mp}(y)$ we must firstly determine the constants $p_{xy}$ and $C_1$ which occur in the equations (8) and (9). The constants are evaluated with the aid of the conditions at $y=\pm1$ and carrying out the similar calculation as Shen we arrive at the following values for $p_{xy}$ and C, ie.

$$C_1 = 0, \qquad (10)$$

$$p_{xy} = \frac{u}{2\sqrt{h\pi}} \Big/ \left\{ \frac{1}{\rho} + \frac{RT}{\mu}\frac{2\sqrt{h}}{\sqrt{\pi}}\frac{1-g_1}{1-g_2} \right\}. \qquad (11)$$

This completes the calculation of the first constant $\alpha_1^{\mp}(y)$. $g_n$'s in (11) are given as

$$g_n = G_n\ (2dk\sqrt{h}) \qquad (n = 0, 1, 2)$$

It is worth remarking that when $\alpha_2^{\mp}(y)$, $\alpha_3^{\mp}(y)$, ....., etc. are taken to be zero in (7), we arrive at the results analogous to those of Shen. In other words the results pertaining to the first term in the series expansion for the velocity distribution function in (7) are similar to those of Shen. As it is impossible to use other transfer equations in Shen's method - consequence of his assumed form of velocity distribution function, the main purpose of the present method is to propose a general form of distribution function so that one can take advantage of the other transfer equations. In the following section we calculate the coefficients of the second term i.e. $\alpha_2^{\mp}(y)$ with the aid of the transfer equations for $m\xi_y$ and $m\xi_y^2$.

*b. Transfer equations for $m\xi_y$ and $m\xi_y^2$; Calculation of $\alpha_2^{\mp}(y)$.*

The transport equation for any $\Phi$ depending on the molecular velocity $\xi$ is written as

$$\nabla_r \int \Phi\ \underline{\xi} f d\underline{\xi} = \int \Phi \left(\frac{\partial f}{\partial t}\right)_{Coll} d\underline{\xi} = n\Delta\bar{\Phi}. \tag{13}$$

expressing it in macroscopic flow quantities we have

$$\begin{aligned} \Phi &= m\xi_j\xi_k \ , \qquad n\Delta\bar{\Phi} = pp_{jk}/\mu_c, \\ \Phi &= \xi_j \frac{m\xi^2}{2} \ , \qquad n\Delta\bar{\Phi} = \left(\frac{p}{\mu_c}\right)(-2q_{j/3} + \Sigma_k p_{jk} U_k), \end{aligned} \tag{14}$$

where $\mu_c$ denotes the classical coefficients of viscosity based on a local full-range maxwellian.

While calculating $\alpha_2^{\mp}(y)$ one also finds the expression for $p_{yy}$ which is not the case with the method of Shen. At this point we differ with the approach of Shen and in this respect it is evident that the present approach is much more general than that of Shen. Further his results come out as a special case of ours.

The transfer equations for $m\xi_y$ and $m\xi_y^2$ turn out to be respectively as

$$\frac{\alpha_2^+}{2\sqrt{\pi}h^{3/2}}(1-G_{3,+}) - \frac{\alpha_2^-}{2\sqrt{\pi}h^{3/2}}(1-G_{3,-}) = \frac{-p_{yy}}{C} - \frac{u}{4h}(G_{2,+} + G_{2,-}), \tag{15}$$

$$\frac{3\alpha_2^+}{8h^2}(1-G_{4,+}) + \frac{3}{8}\frac{\alpha_2^-}{h^2}(1-G_{4,-}) = \frac{RT}{\mu} p_{yy}(y+C_2) + \frac{u}{2\sqrt{\pi}h^{3/2}}(G_{3,-} - G_{3,+}). \tag{16}$$

Once the transfer equations are derived, the rest of the analysis becomes quite simple. At this stage it is apt to remark that more general the distribution functions, the more labour it involves. Carrying out the similar analysis as above one can find the values for $p_{yy}$ and $C_2$. One can calculate the other coefficients by choosing the suitable transfer equations. The method seems to become more and more tedious as one goes to the higher order moments.

## *4. Conclusion.*

The main aim of the present paper is to give a more general formulation for velocity distribution function and in consequence the advantages of the present method are discussed in detail in comparison with other methods, particularly that of Shen. The method is then applied to the problem of plane linearized couette flow and the method of analysis is outlined. Moreover, the method seems to be suitable for non-linear cases. Further work on the applicability of the present method is in progress and it will be published soon.

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